THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and eighth regular meeting of the Society was held at Columbia University on Saturday, February 28, extending through the usual morning and afternoon sessions. The attendance included the following twenty-eight members:

Dr. J. W. Alexander, Professor R. C. Archibald, Professor A. A. Bennett, Professor Pierre Boutroux, Professor B. H. Camp, Professor F. N. Cole, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. H. Gronwall, Professor C. C. Grove, Professor Edward Kasner, Dr. E. A. T. Kircher, Mr. Harry Langman, President E. O. Lovett, Professor H. H. Mitchell, Mr. George Paaswell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. J. M. Stetson, Professor Oswald Veblen, Mr. H. E. Webb, Professor J. H. M. Wedderburn, Professor J. K. Whittemore.

Vice-President R. G. D. Richardson occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. F. J. Burkett, Pennsylvania State College; Mr. A. D. Campbell, Cornell University; Dr. Y. R. Chao, Cornell University; Professor R. E. Gilman, Brown University; Mr. D. C. Kazarinoff, University of Michigan; Dr. Norman Miller, Queen's University; Dr. G. M. Robison, Cornell University; Professor Jung Sun, Peking Academy; Dr. W. H. Wilson, State University of Iowa; Dr. S. D. Zeldin, Massachusetts Institute of Technology. Six applications for membership in the Society were received.

Professor Oswald Veblen was appointed to fill the unexpired term of Professor E. W. Brown, resigned, as representative of the Society in the Division of Physics of the National Research Council. The other representatives are Professor L. E. Dickson, reporter, and Professor H. S. White. These three representatives, the President and the Secretary of the Society, and Professors J. L. Coolidge and L. P. Eisenhart form a committee to consider the formation of an American Section of an International Mathematical Union. Professor Dickson is chairman of this committee.

Steps were taken to submit the question of the incorporation of the Society to the members at the April meeting.

The following papers were read at the February meeting:

(1) Professor JOSEPH LIPKA: "On the general problem of dynamics."

(2) Dr. A. R. SCHWEITZER: "On the iterative properties of the algebra of logic."

(3) Dr. A. R. SCHWEITZER: "On improper pseudogroups, with application to the abstract field."

(4) Mr. G. H. HARDY: "On the representation of numbers as sums of squares and in particular of five and seven."

(5) Dr. J. W. ALEXANDER: "On the representation of any n-dimensional two-sided manifold as a generalized Riemann surface."

(6) Dr. J. W. ALEXANDER: "On the equilibrium of a fluid mass at rest."

(7) Dr. T. H. GRONWALL: "Qualitative properties of the ballistic trajectory (second paper)."

(8) Dr. T. H. GRONWALL: "On the distortion in conformal mapping."

(9) Professor A. A. BENNETT: "Fictitious matric roots of the characteristic equation."

(10) Professor PIERRE BOUTROUX: "On multiform functions defined by differential equations of the first order."

(11) Professor B. H. CAMP: "The significance of a difference, and the value of a sample."

(12) Professor J. H. M. WEDDERBURN: "On division algebras."

(13) Professor EDWARD KASNER: "Geodesics of surfaces and higher manifolds."

Professor Boutroux's paper was a special expository presentation prepared at the suggestion of the programme committee. In the absence of the authors the papers of Professor Lipka, Dr. Schweitzer, Mr. Hardy, and Dr. Alexander's first paper were read by title. Abstracts of the papers follow below.

1. At the annual meeting of the Society, Professor Lipka presented the theorem: if a system of $\infty^{2(n-1)}$ trajectories in space of n dimensions is such that any ∞^{n-1} trajectories of the system which meet an arbitrary hypersurface (space of $n-1$ dimensions) orthogonally admit of ∞^1 orthogonal hypersurfaces, then the system may be considered as the

trajectories in a conservative field of force with a given constant of energy. The purely geometric part of the theorem is true for all natural families of curves. The n-dimensional spaces considered were spaces of constant curvature. In the present paper, the above theorem is demonstrated for spaces of variable curvature. The result is the converse of the Lipschitz theorem announced in *Crelle's Journal*, volume 74 (1871).

2. Given a set S consisting of at least two elements, Dr. Schweitzer shows that this set constitutes an algebra of logic provided the following postulates are satisfied in addition to the property of unique closure with regard to the undefined relations $\theta(x, y)$ between the elements x, y of S:

1. $\theta(x, x) = \theta(y, y)$. 2. $\theta\{x, \theta(y, x)\} = x$.

3. There exists an element 0 such that for every x, y, z, $\theta\{x, \theta(y, z)\} = \psi\theta\{\psi\theta(x, y), \theta[z, \psi(x)]\}$, where $\psi(t) = \theta[0, t]$.

In the second part of the paper iterative relations based on the preceding postulates are given; these relations are mainly in generalization of the relations

$$\theta\{\theta(x, y), \theta(y, z)\} = \theta(x, y),$$

$$\theta\{\theta(y, x), \theta(z, y)\} = \theta(y, x).$$

Concretely, $\theta(x, y)$ may be interpreted by $x +$ not-y.

3. A formal (material) pseudogroup is any system of elements E satisfying the properties $S[\rho_1, \rho_2, \cdots]$ necessary for a formal (material) group and including closure with reference to the undefined relations ρ_i generating S, apart from a set E_0 of exceptional elements. If the set E_0 is empty, then the pseudogroup is proper; otherwise, improper. Dr. Schweitzer proves the following theorem: The abstract field is an improper formal pseudogroup $S[\phi, \psi, \phi_1, \psi_1]$ subject to the condition $\psi_1[x, \phi_1(y, x)] = y$. In case of the abstract group the undefined relations have the interpretations

$$\phi(x, y) = x \cdot y, \qquad \phi_1(x, y) = x^{-1}yx,$$

$$\psi(x, y) = x^{-1} \cdot y, \qquad \psi_1(x, y) = xyx^{-1},$$

and in case of the abstract field the relations may be inter-

preted thus:

$$\phi(x, y) = x + y, \qquad \phi_1(x, y) = x \cdot y,$$

$$\psi(x, y) = -x + y, \qquad \psi_1(x, y) = \frac{1}{x} \cdot y.$$

This result is analogous to a result obtained previously by the author for the algebra of logic.

4. Mr. Hardy's paper contains a detailed development of certain researches of which a short account has been published in volume 4, pages 189–193, of the *Proceedings of the National Academy of Sciences*. The paper will appear in the *Transactions* of the Society.

5. Dr. Alexander proves that any two-sided manifold of n-dimensions may be represented as a generalized Riemann surface spread over a hypersphere.

6. The question was raised by Liapounoff and Poincaré as to whether there exist any figures of equilibrium besides the sphere for a homogeneous fluid mass. In this paper, Dr. Alexander shows that there are no such figures whether of stable or unstable equilibrium.

7. The differential equations of the trajectory being

$$x'' = -G(v)H(y)x', \qquad y'' = -G(v)H(y)y' - g,$$

Dr. Gronwall shows that under fairly broad assumptions on the mode of increase of the functions G and H, the velocity v has only a finite number of maxima and minima as the time t varies from zero to infinity.

8. In the *Comptes rendus de l'Académie des Sciences de Paris*, February 28, 1916, Dr. Gronwall investigated the upper and lower bounds of $|w|$ and $|dw/dr|$, where

$$w = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

maps the circle $|z| < 1$ conformally on a simply connected and nowhere overlapping region in the w-plane, and the coefficient $a_2 = ae^{\gamma i}$ ($a > 0$, γ real) is given a priori. He obtained the exact values of these bounds, except the upper

bound of $|z|$ in the case where $0 \leqq a < 1$, this case being inaccessible to the method used.

In the present paper, it is shown that for any r between zero and unity, and for $0 \leqq a < 1$, we have for $|z| = r$

$$|w| < \frac{1}{4}\left\{\frac{1}{1-a}\left[1-\left(\frac{1-r}{1+r}\right)^{1-a}\right] - \frac{1}{1+a}\left[1-\left(\frac{1-r}{1+r}\right)^{-1-a}\right]\right\},$$

except for the function w obtained by replacing r by $ze^{-\gamma i}$ in the expression to the right, the upper bound being then attained for $z = re^{\gamma i}$.

9. In this paper Professor Bennett shows that many of the simple relations connected with the characteristic equation of a square matrix of order n become obvious if there be assumed to exist n formal matrix quantities (one of which is the given matrix) which are roots of the characteristic equation, commutative with the given matrix, and for which the elementary symmetric functions are scalars. These formal matrices do not always have an actual existence, and their use corresponds in a sense to the introduction of imaginary numbers into algebra.

10. Professor Boutroux's paper first gives a classification of the different types of differential equations belonging to the family $y' + A_0 + A_1y^2 + A_2y^3 = 0$, where the A's are polynomials in x. It then solves, for the simplest type, a problem which has been briefly characterized in the abstract of a paper presented (by title) at the Saint Louis meeting of the Society (see March BULLETIN, pages 271–272).

11. The usual criterion of Tchebycheff for evaluating the probability of the existence of a datum further from the mean value of a distribution than a given multiple of the standard deviation is known to be inadequate. Closer tests are derived by Professor Camp, and applied to the problems of finding the significance of a difference and the value of the mean of a sample. New inequalities used in connection with the point binomial permit a rigorous demonstration of the theorem that, when the distribution from which the

samples are drawn is a point binomial, the distribution of the means may be considered to be the Gaussian law without appreciable error. The point binomial is found to have an invariant property similar to that enjoyed by the Gaussian law.

12. In this paper, Professor Wedderburn shows that the only non-commutative division algebras of order 9 are of the type discovered by L. E. Dickson.

13. Professor Kasner discusses first the ∞^1 geodesics of a surface passing through a given point, giving theorems about the distribution of the osculating spheres. He then studies the ∞^1 plane curves obtained by projecting the geodesics on the tangent plane. These curves all have zero curvature; but J, the rate of variation of curvature with respect to arc, is shown to be a cubic function of the slope. The reconstruction of the surface from the ∞^1 plane curves is then discussed.

For a curved three-dimensional manifold we have ∞^2 geodesics at a point. These are projected on the tangent three-flat, thus giving a bundle of curves in ordinary space. It is shown that these curves have inflexions, the torsion vanishes, but a cubic law is obtained for J. To each tangent line there corresponds an osculating plane; this gives rise to a certain integration problem, and integral cones of the form $X^\alpha Y^\beta Z^\gamma =$ constant are produced, which are recognized as W-cones (analogous to W or anharmonic curves). The correspondence between tangent and binormal for the curves of the bundle is found to be a quadratic Cremona transformation, and is simply visualized and determined by the indicatric quadric surface of the given manifold at the given point. Generalizations for four-dimensional manifolds (required in Einstein's theory) are stated.

Finally the author extends to general manifolds his theorems concerning the limit of the arc to the chord given in this BULLETIN, volume 20 (1914), pages 524–531. This is of importance because when the minimal lines are real, as in the Einstein theory of gravitation, where they appear as light rays, the peculiar limits obtained instead of unity, namely 0.94, 0.86, 0.80, etc., have physical significance.

F. N. COLE,
Secretary.

SOME RECENT DEVELOPMENTS IN THE CALCULUS OF VARIATIONS.*

BY PROFESSOR GILBERT AMES BLISS.

IT is my purpose to speak this afternoon of a part of the theory of the calculus of variations which has aroused the interest and taxed the ingenuity of a sequence of mathematicians beginning with Legendre, and extending by way of Jacobi, Clebsch, Weierstrass, and a numerous array of others, to the present time. The literature of the subject is very large and is still growing. I was discussing recently the title of this address with a fellow mathematician who remarked that he was not aware that there had been any recent progress in the calculus of variations. This was a very natural suspicion, I think, in view of the fact that the attention of most mathematicians of the present time seems irresistibly attracted to such subjects as integral equations and their generalizations, the theory of definite integration, and the theory of functions of lines. It is indeed in these latter domains that the activities especially characteristic of the present era are centered, and the progress already made in them, and the further progress inevitable in the near future, will doubtless be sufficient alone to insure for our generation of mathematical workers a noteworthy place in the history of the science.

While speaking of present day mathematical tendencies I should like to take occasion to mention a remark which has been made to me a number of times by persons who are interested in mathematics primarily for its applications. The feeling of some of these scientists seems to be that mathematical research in America is drawing farther and farther away from the forms of mathematics most immediately useful in related subjects, and they wish to attract the attention and stimulate the interest of mathematicians in the directions of these intermediate domains. We should indeed sympathize heartily with this desire. It is not at all to be regarded

* Address of the retiring chairman of the Chicago Section of the American Mathematical Society, read at the joint meeting of the Chicago and Southwestern Sections, the Missouri Section of the Mathematical Association of America, and Section A of the American Association for the Advancement of Science, at St. Louis, December 30, 1919.

as unfortunate that so much research is being carried on in this country in the purely mathematical field, for one must not forget that the history of mathematics presents repeated instances of pure mathematics of former times which has become applied mathematics of to-day. Furthermore from the educational standpoint it is most important to note that the spirit of research itself, apart from the results attained, is the guarantee in any individual of continuous self-instruction and development, and consequent increase in value to his community. It does seem unfortunate, however, that for some reason the number of people in this country interested in the adaptation of mathematical theories to the pressing contemporary problems of the applications has been relatively small. I am hoping sincerely that cooperation of mathematicians with scientists in other fields will presently be effective in remedying this defect. The mathematical community should assist in every possible way in developing strong applied mathematical research centers especially where beginnings have already been made, in encouraging graduate students to investigations in the fields intermediate between mathematics and the neighboring sciences, and in the discovery and dissemination among maturer workers of information concerning the applied mathematical problems now awaiting solution. In this connection let me also suggest that mathematicians would do well to inquire searchingly into the advantages or disadvantages of cooperative work among themselves, as well as with scientists in other fields. Cooperation seems to be the watchword of the time, and we should not be the last to recognize it. I am myself not yet convinced of the economy of group effort as over against individual effort in mathematical research, but I believe it important that in the near future we should have some conclusive information about it.

Let me return, however, to the original topic of my discourse which was to have been a part of the theory of the calculus of variations. I should like very much to dwell on some of the modifications which have been made possible by recent progress in the domains of real function theory mentioned in my first paragraph above, but it would be impossible in the limited time at my disposal. Instead I wish to speak of a new method of treating the second variation in the calculus of variations, a method which has rested partly developed

in my mind for some time, and which has recently turned out to be effective in simplifying the theory of the second variation and coordinating the complicated and extensive literature of the subject.

The simplest problem of the calculus of variations, the one to which my expository remarks this afternoon must be for the most part restricted, is that of determining a curve

$$(C) \qquad y = y(x) \quad (x_1 \leqq x \leqq x_2)$$

in the xy-plane, joining two fixed points (x_1, y_1) and (x_2, y_2), and minimizing an integral

$$(1) \qquad J(C) = \int_{x_1}^{x_2} f(x, y(x), y'(x))dx.$$

In order to obtain conditions which must be satisfied by a minimizing arc let us consider the values of the integral along the curves of a family of the form

$$\bar{y} = y(x) + \alpha\eta(x) \quad (x_1 \leqq x \leqq x_2),$$

where α is a constant to be varied at pleasure and $\eta(x)$ is a function which vanishes at x_1 and x_2, so that $\eta(x_1) = \eta(x_2) = 0$. All of the curves of this family pass through the end points of the original curve C, on account of the last condition imposed upon the $\eta(x)$, and it is clear that the function of α,

$$J(\alpha) = \int_{x_1}^{x_2} f(x, y + \alpha\eta, y' + \alpha\eta')dx,$$

must have a minimum for $\alpha = 0$, so that by the usual theory of maxima and minima the conditions

$$(2) \qquad J'(0) = \int_{x_1}^{x_2} (f_y\eta + f_{y'}\eta')dx = 0,$$

$$(3) \qquad J''(0) = \int_{x_1}^{x_2} (f_{yy}\eta^2 + 2f_{yy'}\eta\eta' + f_{y'y'}\eta'^2)dx$$

$$= \int_{x_1}^{x_2} 2\Omega(x, \eta, \eta')dx \geqq 0$$

must be satisfied for every choice of the function $\eta(x)$ vanishing at x_1 and x_2. Here the arguments of the derivatives of f are the $y(x)$ and $y'(x)$ which belong to the curve C, and the symbol

$2\Omega(x, \eta, \eta')$ is merely a notation for the quadratic form which appears in the integrand of $J''(0)$. The expressions (2) and (3) are called, respectively, the first and second variations of the integral J along the arc C.

The first term of the first variation can be integrated by parts in simple fashion giving the equation

$$J'(0) = \int_{x_1}^{x_2} \left(f_{y'} - \int_{x_1}^{x} f_y dx \right) \eta' dx = 0$$

which must hold for every $\eta(x)$ vanishing at x_1 and x_2, and it is provable from this condition that there must be a constant c such that along the minimizing arc C

$$f_{y'} = \int_{x_1}^{x} f_y dx + c. \tag{4}$$

This is a very important equation from which can be deduced several consequences. In the first place the derivative $f_{y'}$ must be continuous all along the minimizing arc C, since the second member of the equation has this property, so that at each corner point (x_0, y_0) of C the equation

$$f_{y'}(x_0, y_0, y'(x_0 - 0)) = f_{y'}(x_0, y_0, y'(x_0 + 0)) \tag{5}$$

must be satisfied by the two slopes $y'(x_0 - 0)$, $y'(x_0 + 0)$ of the arc C toward the right and left from the corner. In the second place it is provable at once by differentiating the two sides of the equation that the differential equation

$$\frac{d}{dx} f_{y'} = f_y \tag{6}$$

must be satisfied everywhere on C. Finally it follows from the equation (4) by somewhat more complicated reasoning that the minimizing arc, supposed at first to have only a first derivative y' elsewhere than at corners, must have also higher derivatives, at least at every point at which the condition $f_{y'y'} \neq 0$ is satisfied, i.e. at every point at which the differential equation (6) is non-singular.

The differential equation (6) of the minimizing arc is now usually ascribed to Euler,* though for a long time it was called

* Methodus inveniendi, etc. (1744), Chapter II, Section 21; translated into German in Ostwald's Klassiker der exakten Wissenschaften, No. 46, p. 53.

the Lagrange equation, because Lagrange adopted it and used it so widely. The integration by parts which I have mentioned in treating the first variation is due to du Bois Reymond.* The corner condition (5) was deduced by Weierstrass for the parametric case in his lectures as early as 1865, but was rediscovered by Erdmann in 1877† before the results of Weierstrass were published. The differentiability properties of the minimizing arc were studied by Hilbert‡ by means of the equation (4), but I think that the derivation of them from this equation by a very simple application of implicit function theorems was due to Mason and myself.§

Before proceeding to the theory of the second variation which is my ultimate goal, I must mention two further contributions to the theory by Weierstrass, familiar to those of you who have studied the calculus of variations, but perhaps not so to others. In his lectures of 1879 Weierstrass introduced his so-called E-function

$$E(x, y, y', \bar{y}') = f(x, y, \bar{y}') - f(x, y, y') - (\bar{y}' - y')f_{y'}(x, y, y').$$

It is clearly the difference of $f(x, y, \bar{y}')$ and the sum of the first two terms obtained by expanding this function by Taylor's formula in powers of $\bar{y}' - y'$. With the help of very simple types of comparison curves he established the further condition that

$$E(x, y, y', \bar{y}') \geqq 0$$

at every set of elements $(x, y, y', \bar{y}')$ for which x, y, y' belongs to the arc C and $\bar{y}'$ is arbitrary. This was a new and important result, for all the necessary conditions deduced before his time had involved only comparison curves which hugged closely the original arc C both in position and direction, and he was the first to show that the admission of variations which departed widely in direction, though not in position, from the minimizing arc C gives rise to this further restrictive property.

But he added to the theory in other important respects also, only one of which I can here lay before you. The equation (6), after the differentiation with respect to x has

* *Math. Annalen*, vol. 15 (1879), p. 313.
† *Journal für Mathematik*, vol. 82 (1877), p. 21.
‡ *Göttinger Nachrichten*, 1900, p. 253.
§ "The properties of curves in space which minimize a definite integral," *Trans. Amer. Math. Society*, vol. 9 (1908), p. 440.

been performed, has the form

$$f_{y'x} + f_{y'y}y' + f_{y'y'}y'' = f_y, \qquad (6)$$

which shows it to be of the second order, and its solutions form a two-parameter family of curves which are called the extremals of the problem. Weierstrass considered especially one-parameter families of such extremals of the form

$$y = \varphi(x, a), \qquad (7)$$

containing the particular arc C, whose minimizing properties are under investigation, for a particular parameter value a_0. If a family of this sort can be found which simply covers a neighborhood F of C, then the region F with its covering of extremals is called a field about C. By saying that F is simply covered by the family (7) it is meant that through each point of the region there passes one and but one extremal of the family, so that with each point (x, y) of the field F there is associated a unique extremal slope $p(x, y)$. If now $\bar{C}$ is an arbitrary curve

$$y = \bar{y}(x) \quad (x_1 \leqq x \leqq x_2) \qquad (\bar{C})$$

in the field F joining the end points of C then Weierstrass showed that the difference $J(\bar{C}) - J(C)$ of the values of J along the two arcs is expressible by the formula

$$J(\bar{C}) - J(C) = \int_{x_1}^{x_2} E(x, \bar{y}, p(x, \bar{y}), \bar{y}')dx \qquad (8)$$

in which $\bar{y}$ stands always for the function $\bar{y}(x)$ defining the arc $\bar{C}$.

It is clear from this formula that $J(C)$ will surely be a minimum if the arc C can be imbedded in a field F of the type described above, and if the integrand in the second member of the formula (8) is always positive along arcs of the type of $\bar{C}$ in the field. I should be giving a resumé of most of the theory of the calculus of variations if I should attempt to describe in detail the conditions under which the field F can be constructed, and which insure the minimizing properties of C. They are, however, well known, and for this simple case are not difficult to develop and present.

Hilbert* devised a very elegant method of establishing the formula (8) which has now superseded with most writers the original "Weierstrass construction" by means of which Weierstrass himself established his result. Hilbert showed that the value of the integral

$$J^*(\bar{C}) = \int_{x_1}^{x_2} \{f(x, \bar{y}, \bar{y}') + [\bar{y}' - p(x, \bar{y})] f_{y'}(x, \bar{y}, p(x, \bar{y}))\} dx$$

is independent of the path $\bar{C}$ chosen in the field joining two fixed points, and that along an extremal arc of the field $J^*(C) = J(C)$, since along such an arc the differential equation $y' = p(x, y)$ is always satisfied. It follows then at once that

$$J(\bar{C}) - J(C) = J(\bar{C}) - J^*(C) = J(\bar{C}) - J^*(\bar{C})$$

$$= \int_{x_1}^{x_2} E(x, \bar{y}, p(x, \bar{y}), \bar{y}') dx.$$

From a very early period in the history of the calculus of variations to the time of Weierstrass it was thought that an arc C which made the first variation zero and the second variation positive for all choices of the function η, would surely minimize the integral J. The acceptance of this assumption, unjustified by any proof, was due to the analogy between the calculus of variations and the theory of maxima and minima of ordinary functions of ordinary variables. The assumption is correct in the case of a weak minimum, for which the only comparison curves admitted are those which are near to C both in position and in direction, but Weierstrass showed that it is incorrect in the case of a strong minimum, for which the comparison curves are required to be close to C in position only. He developed his necessary E-function condition, in addition to those which are deduced from the first and second variations, and proved that the conditions so found when slightly modified are sufficient to insure the existence of a minimum along the arc C.

One result of this partial misconception by those who preceded Weierstrass was an elaborate theory of the second variation, the purpose of which was to reduce it to a form in which it would be clearly positive for all choices of the function

* An exposition is given by Bolza, Vorlesungen über Variationsrechnung, pp. 105ff.

$\eta(x)$. Legendre, in 1786,* was the first to attempt such a transformation. Euler, Lagrange, and other investigators before him had obtained from the first variation the differential equation of the minimizing curves, and conditions at the ends of the curves when the end points were allowed to vary. He himself begins his paper rather apologetically, it seems to me, his avowed purpose being to exhibit a criterion for distinguishing between extremal arcs which furnish maxima and those which furnish minima, whereas he says that in a majority of the applications one can usually readily decide this question from the setting of the problem itself. Inspired, however, by the analogous theory of maxima and minima of functions of a finite number of variables, he succeeded in transforming the second variation into the form

$$J''(0) = \int_{x_1}^{x_2} f_{y'y'}\left(\eta' + \frac{f_{yy'} + \alpha}{f_{y'y'}}\eta\right)^2 dx,$$

where α is a solution of a certain Riccati differential equation of the first order. From this he concluded that the second variation will always be positive and therefore that the integral (1) will always be minimized by an extremal arc C along which $f_{y'y'} > 0$. His proof was faulty because there does not always exist a function α of the type he used, finite everywhere between the limits x_1 and x_2, but his conclusion that the positive form found for the second variation would ensure a minimum is correct for the case of a weak minimum, as I have already indicated. It turns out that the condition, $f_{y'y'} \geqq 0$ along the minimizing extremal arc, is necessary for a minimum but not always sufficient, and it is now called the Legendre necessary condition. It is interesting to note that Legendre himself was not entirely satisfied with his analysis, for he published later comments on his own paper in which he called attention to some of the weaknesses of his argument and suggested methods of overcoming them.

In 1837 Jacobi† announced the discovery of an important advance in the theory of the transformation of the second variation. He disclosed the circumstances under which the trans-

* "Abhandlung über die Untersuchung der Maxima und Minima in der Variations-Rechnung," *Mémoires de l'Académie royale des Sciences*, 1786, p. 7.

† "Zur Theorie der Variationsrechnung und der Differentialgleichungen," *Journal für die reine und angewandte Mathematik*, vol. 17 (1837), p. 68.

formation of Legendre was possible and characterized the cases when it is not possible. On every extremal arc with a fixed initial point A there is in general a point A' conjugate to A such that Legendre's transformation is possible on every arc AB excluding A', but not possible on an arc AB containing A'. In justifying these statements Jacobi introduced, in place of the Riccati equation used by Legendre, a linear differential equation of the second order, now usually called the Jacobi differential equation, and he showed that its solutions can be obtained from those of Euler's equation by differentiations with respect to the constants of integration. He also used a new method for transforming the second variation which he applied successfully to the case when the integrand of the integral (1) contains higher derivatives of y. His conclusions were that an extremal arc AB which contains the conjugate point A' either between A and B or at B can not minimize the integral J, and that an extremal arc AB which has $f_{y'y'} > 0$ along it and which does not contain the conjugate point A' will surely minimize J.

Jacobi's results were correct with the exception of his statement that no minimum can exist when the point A' conjugate to A coincides with B. It is now known that in this exceptional case there will sometimes be a minimum and sometimes not. But his proofs were again incomplete. I do not find in examining his paper that he justifies anywhere the necessity of the condition that the conjugate point A' shall not lie between A and B, and his sufficiency proof rests solely upon his transformation of the second variation to positive form and is open to the same objection which I have mentioned before as applying to all of the writers who preceded Weierstrass.

In extending his transformation to integrals containing higher derivatives of the function $y(x)$ Jacobi made use of a theorem concerning linear differential equations which he did not prove. It is closely associated with the now well-known relation

$$u\,J(v) - v\,J(u) = \frac{d}{dx}M(u, v)$$

satisfied by a self-adjoint linear differential equation of the nth order. His incomplete exposition aroused much interest and led to a series of papers by Delaunay, Minding, Spitzer,

Heine, Clebsch, Hesse, Lipschitz, Mayer, Scheffers, von Escherich, and others, who gave proofs of the Jacobi theorem and extended his results to more complicated problems of the calculus of variations. Of these papers I shall mention especially only those which seem to me to contain the more important advances.

In 1858 Clebsch* studied the transformation of the second variation for the very general type of problem formulated by Lagrange. For such a problem there is an integral of the form

$$\int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx,$$

to be minimized in the class of curves

$$y_i = y_i(x) \quad (x_1 \leqq x \leqq x_2\,;\, i = 1, \cdots, n)$$

which join two fixed points and which satisfy besides a set of differential equations

$$\varphi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \quad (i = 1, \cdots, m < n).$$

Clebsch found the system of linear differential equations of the second order analogous to the single Jacobi equation in the simpler case which we have been considering, and he expressed the second variation as an integral

$$\int_{x_1}^{x_2} \sum_{i,k} F_{y_i' y_k'} W_i W_k dx. \tag{9}$$

In this expression F is the sum

$$F = f + \lambda_1\varphi_1 + \cdots + \lambda_m\varphi_m,$$

the coefficients λ_i being functions of x, and the variables W_i are proportional to certain determinants of order $n + 1$ which satisfy the relations

$$\frac{\partial \varphi_i}{\partial y_1'} W_1 + \cdots + \frac{\partial \varphi_i}{\partial y_n'} W_n = 0 \quad (i = 1, \cdots, m).$$

The Jacobi transformation of the second variation for the simpler problems is possible if there is a solution u of

* *Journal für die reine und angewandte Mathematik*, vol. 55 (1858), pp. 254 and 335; in two papers: "Ueber die Reduktion der zweiten Variation auf ihre einfachste Form," "Ueber diejenigen Probleme der Variationsrechnung, welche nur eine unabhängige Variable enthalten."

Jacobi's equation which does not vanish on the interval x_1x_2. For the Lagrange problem here under consideration there must be an n-square matrix with columns belonging to n so-called conjugate systems of solutions of the Jacobi equations and having a determinant different from zero on x_1x_2.

The transformation of Clebsch seems to me a very complicated one. It is based upon a clever foresight of the form to which the integrand of the second variation may be reduced, and involves a count of the conditions necessary to make the reduction, the introduction of a sufficient number of arbitrary functions to satisfy these conditions, and the determination of the functions so introduced by a sequence of systems of equations. From the results of the transformation one may infer that the second variation will be positive whenever the quadratic form in the integrand of the expression (9) is positive at each point of the extremal arc under consideration. It is also provable that this property is necessary for a minimum, and the condition so found is sometimes called the Legendre-Clebsch condition because it is the generalization of the one which Legendre deduced for the simpler case.

Ten years after the papers of Clebsch appeared A. Mayer* presented the same results in somewhat simpler form and extended the Jacobi theory of the conjugate point to fit the Lagrange problem. He determined the position of the conjugate point by means of a zero of a determinant of solutions of Jacobi's equations, now called the Mayer determinant.

But the most thorough study of the theory of the transformation of the second variation is undoubtedly that of von Escherich.† In a series of five papers in 1898, 1899, and 1901 he effected the Clebsch transformation for several types of problems by a generalization of Jacobi's method, and studied in great detail the theory of conjugate points and the conjugate systems of solutions of Jacobi's equations which are important in the determination of these points as well as in the transformation itself. The equations which I am designating here as the Jacobi equations, because they are generalizaitons of

* "Ueber die Kriterien des Maximums und Minimums der einfachen Integrale," *Journal für die reine und angewandte Mathematik*, vol. 69 (1868), p. 238.

† "Die zweite Variation der einfachen Integrale," *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften in Wien*, Mathematisch-naturwissenschaftliche Classe, vols. 107, 108, 110 (1898, 1899, 1901).

the one which Jacobi used in the simpler case, are called by von Escherich the accessory system of linear differential equations. I will mention especially, without attempting to write it down, an important formula which lies at the basis of the von Escherich theory. It is a formula involving the determinants of conjugate systems and is of great assistance in the discussion of the zeros of the Mayer determinant which determines conjugate points. It should be added that von Escherich discovered, as a result of his detailed study, an important exceptional case for the Lagrange problem which he distinguishes from the more general normal or non-singular type. Most of his results apply only to the latter, and little is in fact known of the possibilities of the singular case.

A discussion of the second variation could not be complete without mentioning the work of Weierstrass. His results were obtained between 1865 and 1888 but they became known very slowly because he described them only in his lectures. The book by Kneser, Vorlesungen über Variationsrechnung, 1900, was I think the first published account of any importance. Weierstrass confined his attention for the most part to the problem in parametric form in the plane. His transformation of the second variation for this case was exceedingly ingenious, but apparently not readily generalizable to higher spaces. His other work, however, had also an important influence upon the study of the second variation. For the sufficiency proof which he developed for the case of a strong minimum applies also to a weak minimum, and a sufficiency proof in the latter case based on the theory of the transformation of the second variation is therefore superfluous.

Finally I should mention an interesting paper by Hahn* in which he shows the connection between the conjugate systems of solutions of Jacobi's equations and what are called Mayer fields. These latter are generalizations of the plane fields of which I have spoken above. For problems in higher spaces the analogue of the field covered by a one-parameter family of solutions of Euler's equations is an n-parameter family of extremals simply covering a region F of space. But the Hilbert integral will be invariant in such a region F only if the n-parameter family itself has special properties, in which case F with its extremals is called a Mayer field

* "Ueber den Zusammenhang zwischen den Theorien der zweiten Variation und der Weierstrass'schen Theorie der Variationsrechnung," *Rendiconti del Circolo Matematico di Palermo*, vol. 29 (1910), p. 49.

because Mayer was the first to recognize and study the properties of such families. The phenomena presented by the n-parameter families of Mayer fields are analogous to those of two-parameter families of straight lines in space. Not every such family is orthogonal to a surface. When one has this property it will form a Mayer field for the problem of minimizing the ordinary length integral in space. It turns out that every conjugate system of solutions of the Jacobi equations is deducible from the equations of the n-parameter family of a Mayer field by differentiation with respect to its parameters, and Hahn uses this property to assist in constructing the various types of conjugate systems which von Escherich studied.

I have now come to the results of which I wish especially to speak this afternoon. The historical sketch which I have given, though most incomplete, will suffice to show the great interest which mathematicians have taken in the theory of the second variation, and the complicated character of the transformations which have been devised for it. The results of Weierstrass mentioned above, and some very beautiful geometrical methods, developed by Darboux for geodesics and generalized by Kneser, enable one to deduce the necessary conditions of Legendre and Jacobi in some cases without the use of the complicated theory of the second variation. These proofs fail, however, when the enveloping curve upon which they depend has a singularity of a special type, and the efforts which have been made to complete the proofs to cover all cases have led to complications as great as those of the theory of the second variation itself.

The importance of the theory of the second variation at the present time lies therefore primarily in the proofs which it provides for the necessity of the analogues, for more general problems, of the conditions discovered by Legendre and Jacobi for the simpler cases. Furthermore the transformation of the second variation into a form clearly positive may be made the basis for the justification of a set of sufficient conditions for a weak minimum, and if the transformation can be effected simply the method will be an economical one. Finally I regard it as highly desirable from a historical point of view that the complicated theories which have been attached to the second variation should have an interpretation, in the light of recent results in the calculus of variations, which will make them more

perspicuous and less detached than they seem to be at present from the theory as a whole.

For some years past I have been in the habit, in my lecture courses at the University of Chicago, of deducing the Legendre and Jacobi conditions for simpler problems by an application to the second variation of results described in the earlier part of this paper, the first two necessary conditions mentioned as deducible from the first variation and the Weierstrass E-function condition. The underlying thought is this. Since the second variation must always be positive or zero along a minimizing arc for all choices of the function $\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$, we are led to the consideration of a minimum problem in the $x\eta$-space for the integral (3) of the second variation. This problem is of precisely the same sort as the original problem of minimizing the integral (1). The curve $\eta = 0$ is necessarily a minimizing curve for the $x\eta$-problem, when $J(C)$ is a minimum, since it gives the second variation its smallest value zero. Hence along it we must have the E-function condition satisfied,

$$\Omega(x, \eta, \bar{\eta}') - \Omega(x, \eta, \eta') - (\bar{\eta}' - \eta')\Omega_{\eta'}(x, \eta, \eta') = f_{y'y'}(\bar{\eta}' - \eta')^2 \geqq 0,$$

the first member of the equality being the Weierstrass E-function for the integrand function Ω of the second variation. It follows at once that the Legendre condition $f_{y'y'} \geqq 0$ must hold along the minimizing arc.

Furthermore, the Euler differential equation of this minimum problem of the second variation is the equation

$$\Omega_\eta - \frac{d}{dx}\Omega_{\eta'} = (f_{yy}\eta + f_{yy'}\eta') - \frac{d}{dx}(f_{y'y}\eta + f_{y'y'}\eta') = 0$$

which is precisely the linear differential equation of the second order in η upon which Jacobi based his theory of the second variation. The coefficient of η'' in this equation is $f_{y'y'}$, and in discussing the Jacobi condition it is customary to presuppose this coefficient different from zero along the arc C, so that the solutions of the Jacobi equation will have no singularities on the interval x_1x_2. Suppose now that there is a solution u of Jacobi's equation not identically zero but vanishing at x_1 and at a second point x_3 between x_1 and x_2. Then in the first place the value $u'(x_3)$ is different from zero, since the

only solution of Jacobi's equation vanishing with its derivative at any point whatsoever is $\eta \equiv 0$. Furthermore the function $\eta(x)$ defined by the conditions

$$\eta(x) = u(x) \quad \text{for} \quad x_1 \leqq x \leqq x_3,$$
$$\eta(x) = 0 \quad \text{for} \quad x_3 \leqq x \leqq x_2,$$

is an admissible function η. It is easily provable that $\eta(x)$ gives the second variation its minimum value zero, for with the help of an integration by parts and properties of the homogeneous quadratic function Ω we have

$$J''(0) = 2\int_{x_1}^{x_3} \Omega(x, u, u')dx = \int_{x_1}^{x_3} (u\Omega_u + u'\Omega_{u'})dx$$
$$= u\Omega_{u'}\Big|_{x_1}^{x_3} + \int_{x_1}^{x_3} u\left(\Omega_u - \frac{d}{dx}\Omega_{u'}\right)dx = 0.$$

The curve defined by $\eta(x)$ can not, however, be a minimizing curve for the second variation since it has a corner at x_3 at which

$$\Omega_{\eta'}(x_3 - 0) - \Omega_{\eta'}(x_3 + 0) = f_{y'y'}u'\Big|^{x_3} \neq 0.$$

Since $\eta(x)$ gives the second variation the value zero and does not minimize it, we conclude that there are variations $\bar{\eta}(x)$ giving the second variation a value less than zero, and it follows at once that the original integral J can not be minimized by the arc C. We have then the result that if $J(C)$ is a minimum there is no solution $u \not\equiv 0$ of Jacobi's equation vanishing at x_1 and at a second point x_3 between x_1 and x_2. This is, however, exactly Jacobi's necessary condition, since the conjugate point to $A(x = x_1)$ is the point $A'(x = x_3)$ corresponding to the first zero x_3 of $u(x)$ after x_1, and we see that arcs AB containing A' between A and B have no longer the minimizing property.

It is to be noted that the proofs of the necessity of the Legendre and Jacobi conditions just described are made under the hypotheses usually presupposed when these conditions are deduced from the second variation, and I wish further to call attention to the fact that they do not involve any transformation of the second variation except the very simple one used in showing that the curve $\eta(x)$ makes the second variation vanish. The Euler differential equation condition, the corner

point property, and the Weierstrass E-function condition, are properties of a minimizing arc which should be developped early in every complete discussion of the problem, and when we apply them to the second variation we are not using new methods peculiar to this later part of the theory. The effectiveness and economy of the methods just described are not as important for the simple case, which has been discussed in some detail, as they are for the more complicated problems of the calculus of variations. D. M. Smith* has, however, extended these methods so that they can be applied to the problem of Lagrange, and Miss G. A. Larew† has done the same for the problem of Mayer. I have myself modified them for problems in parametric form in higher spaces.‡ In these latter cases a transformation of the second variation analogous to that of Weierstrass for the plane seems now entirely unnecessary.

But it has seemed to me for some time that the theory of the second variation in its entirety could be viewed with success from the standpoint of the minimum problem of the second variation, a minimum problem within a minimum problem, so to speak, and I have recently satisfied myself that this is so for the Lagrange problem as well as for the simpler cases. A one-parameter family of extremals of the second variation simply covering a field in the $x\eta$-plane is a family of solutions of Jacobi's equations of the form

$$\eta = au \quad (x_1 \leqq x \leqq x_2), \tag{10}$$

where u is a particular solution of the Jacobi equation which does not vanish on the interval x_1x_2, and a is the parameter of the family. The slope function $\pi(x, \eta)$ for this field, expressing the slope of the extremal of the field through the point (x, η) and corresponding to the slope function $p(x, y)$ of the general case, is found by solving the last equation for a and substituting in the expression

$$\pi(x, \eta) = au' = \eta \frac{u'}{u}.$$

* "Jacobi's condition for the problem of Lagrange in the calculus of variations," *Trans. Amer. Math. Society*, vol. 17 (1916), p. 459.

† "Necessary conditions in the problems of Mayer in the calculus of variations," *Trans. Amer. Math. Society*, vol. 20 (1919), p. 1.

‡ "Jacobi's condition for problems of the calculus of variations in parametric form," *Trans. Amer. Math. Society*, vol. 17 (1916), p. 195.

If $J''(H)$ represents the value of the second variation for an arbitrary function $\eta(x)$ vanishing at x_1 and x_2, and if $J''(H_0)$ is the corresponding value along $\eta = 0$, then the formula (8) of Weierstrass when applied to the second variation takes the form

$$J''(H) - J''(H_0)$$

$$= \int_{x_1}^{x_2} \{\Omega(x, \eta, \eta') - \Omega(x, \eta, \pi) - (\eta' - \pi)\Omega_{\eta'}(x, \eta, \pi)\}dx.$$

But since $J''(H_0) = 0$, an application of Taylor's formula to the integrand on the right gives

$$J''(H) = \int_{x_1}^{x_2} f_{y'y'}(\eta' - \pi)^2 dx = \int_{x_1}^{x_2} f_{y'y'}\left(\frac{\eta' u - \eta u'}{u}\right)^2 dx.$$

This is precisely the form of the second variation found by Jacobi. From it we conclude at once that the second variation is positive or zero for every function η vanishing at x_1 and x_2, provided that $f_{y'y'} > 0$ along the original arc C, and that a solution u of Jacobi's equation exists which is different from zero on the interval x_1x_2.

The analysis in the case of the Lagrange problem is of course more complicated. The one-parameter family (10) of the field for the second variation is here replaced by an n-parameter family

$$\eta_i = a_1u_{i1} + \cdots + a_nu_{in} \quad (i = 1, \cdots, n), \tag{11}$$

for which the columns of the matrix $||u_{ik}||$ belong to a conjugate system of solutions of the Jacobi equations. The slope functions $\pi_i(x, \eta)$ of the field are found by solving the last equations for the parameters a_i and substituting them in the expressions

$$\pi_i = a_1u_{i1}' + \cdots + a_nu_{in}' \quad (i = 1, \cdots, n),$$

where u_{ik}' is the derivative of u_{ik} with respect to x. The Weierstrass formula gives at once the value

$$J''(H) = \int_{x_1}^{x_2} \sum_{ik} F_{y_i'y_k'}(\eta_i' - \pi_i)(\eta_k' - \pi_k)dx$$

for the second variation, where F has the significance before explained, and where the differences $\eta_i' - \pi_i$ turn out to

have the values

$$\eta_i' - \pi_i = W_i = \frac{1}{|u_{ik}|}\chi_i(\eta)$$

in terms of the functions W_i of Clebsch and the determinants $\chi_i(\eta)$ of von Escherich. These latter functions have hitherto appeared in somewhat artificial fashion in the theory and it seems to me interesting to have this very simple interpretation of them in terms of the slope functions of the field.

The von Escherich fundamental formula which I mentioned above* may be expressed by the equation

$$\sum_{ik} F_{\eta_i'\eta_k'}(\eta_i' - \pi_i)(\eta_k' - \pi_k) = \sum_i a_i'\psi_i,$$

where the symbols a_i' represent the derivatives of the solutions a_i of the equations (11) with respect to x when the variables η_i are thought of as functions of x, and the coefficients ψ_i are bilinear expressions in the elements of two sets of functions $(\eta_1, \cdots, \eta_n;\ \mu_1, \cdots, \mu_m)$ and $(u_{1i}, \cdots, u_{ni};\ \sigma_{1i}, \cdots, \sigma_{mi})$, the latter of which is the system of solutions of Jacobi's equations to which the column $u_{1i}, \cdots, u_{ni}$ of the matrix $||u_{ik}||$ belongs. If the set $(\eta;\ \mu)$ is also a solution of Jacobi's equations then each of these coefficients is a constant. For this special case I have already published a proof† of the von Escherich formula, but the one which I have mentioned here is quite general and simpler than my earlier one.

Finally the von Escherich theory of conjugate systems of solutions of Jacobi's equations is identical with the theory of Mayer fields for the minimum problem of the second variation, and the methods which Hahn devised for relating conjugate systems to the Mayer fields of the original minimum problem can be used to construct the various types of conjugate systems which von Escherich considered.

In conclusion I may say that the methods suggested above seem to me of considerable assistance in developing and understanding the theory of the calculus of variations. For in the first place they enable one to retain the advantages of the second variation, in deducing the necessary conditions analogous to those of Legendre and Jacobi, without becoming involved in elaborate transformations. In the second place

* Von Escherich, loc. cit., Mittheilung IV, Section XXIII, formula (9).

† "A note on the problem of Lagrange in the calculus of variations," this BULLETIN, vol. 22 (1916), p. 220.

they may give some satisfaction to those who have delved in the theory of those transformations, since they make the transformations appear, not as separate chapters of the analysis, but as special applications of formulas now well established in other parts of the theory. It is even possible to make a satisfactory and not over complicated sufficiency proof for a weak minimum, without the use of the Weierstrass notion of a field. For the Weierstrass formula can be proved directly with some ease for the second variation when once it has been seen to hold true for a conjugate system of solutions of Jacobi's equations. It is not strange that the second variation has not been attacked from this standpoint before in spite of the fact that in my recent review of the literature I have found several suggestions which might have instigated one to attempt it. The real reason is, I think, that the advances of Weierstrass and Hilbert were published after 1900 and about the time that Kneser found his envelope theorem. The tendency since then has been to discard the theory of the second variation in favor of the more geometrical theory, but the experiment, so far as I know, has not been completely successful.

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GROUPS GENERATED BY TWO OPERATORS OF ORDER THREE WHOSE PRODUCT IS OF ORDER FOUR.

BY PROFESSOR G. A. MILLER.

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§ 1. *Introduction.*

IT is known that the only groups which are completely determined by the orders of two generators and the order of their product are the dihedral groups and the groups of movements of the five regular solids known to the ancients. In all other cases two generators which are not restricted except as regards their orders and the order of their product

may give rise to any one of an infinite system of groups.* In particular, there is no upper limit to the number of distinct groups that can be generated by two operators of order three whose product is of order four. The group of smallest order which two such operators can generate is the well known group of order 24 which does not contain any subgroup of order 12, and it is known that this group is defined by the facts that it is generated by two operators of order 3 whose product is of order 4 provided that the square of this product is invariant under each of these two operators of order 3.

In the present article we shall consider the groups generated by s_1 and s_2 when these operators satisfy the following conditions:

$$s_1^3 = s_2^3 = (s_1s_2)^4 = (s_1^2s_2)^k = 1 \quad (k = 3, 4, 5).$$

It is unnecessary to consider the case where $k = 2$, since the equations $s_1^3 = s_2^3 = (s_1^2s_2)^2 = 1$ define the tetrahedral group and hence s_1s_2 could not be of order 4. When $k = 1$ the order of s_1s_2 could evidently not be 4. Hence 3, 4, 5 are the smallest positive integral values of k when s_1s_2 is of order 4.

§ 2. *The Order of $s_1^2s_2$ is Three.*

From the condition that $(s_1^2s_2)^3 = 1$ it follows that

$$s_1^2s_2s_1^2 = s_2^2s_1s_2^2, \qquad s_1s_2^2s_1 = s_2s_1^2s_2.$$

Hence the two operators of order 4, s_1s_2 and s_2s_1, are commutative and generate a group whose order cannot exceed 16. This group must be invariant under the group G generated by s_1 and s_2 since

$$s_1^2s_2s_1^2 = s_1^2s_2^2 \cdot s_2^2s_1^2 \text{ and } s_2^2s_1s_2^2 = s_2^2s_1^2 \cdot s_1^2s_2^2.$$

The order of G can therefore not exceed 48. That the order of G cannot be less than 48 when no additional restrictions are imposed on s_1 and s_2 results directly from the fact that the abelian group of order 16 and of type (2, 2) admits an automorphism of order 3 in which no operator except identity corresponds to itself. Hence the theorem:

If two operators s_1, s_2 satisfy the conditions

$$s_1^3 = s_2^3 = (s_1^2s_2)^3 = (s_1s_2)^4 = 1$$

but are not otherwise restricted, they generate the group of order

* G. A. Miller, *Amer. Journal of Mathematics*, vol. 24 (1902), p. 96.

48 obtained by extending the abelian group of order 16 and of type (2, 2) by means of an operator of order 3 which transforms this group into itself but is not commutative with any of its operators besides identity.

This group of order 48 evidently contains 32 operators of order 3, 12 operators of order 4, and 3 operators of order 2 in addition to identity. Each operator of order 3 is commutative with its own powers only, while each of the other operators besides identity is commutative with exactly 16 operators of the group.

§ 3. *The Order of $s_1^2 s_2$ is Four.*

It is easy to see that when $s_1^2 s_2$ is of order 4 the order of G is divisible by 168 since the two substitutions $s_1 = abc \cdot def$, $s_2 = aeh \cdot cdg$ satisfy the given conditions, as results from the following equations:

$$s_1 s_2 = abdh \cdot cefg, \qquad s_1^2 s_2 = adfh \cdot begc.$$

These two substitutions generate a transitive substitution group of degree seven whose order is a multiple of 168 since $s_1^2 s_2 s_1 s_2^2 = bfedcgh$. The fact that the order of $s_1^2 s_2 s_1 s_2^2$ is actually 7 when s_1, s_2 represent two abstract operators which satisfy the conditions

$$s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1^2 s_2)^4 = 1$$

but are not otherwise restricted results from the following equations:

$$(s_1^2 s_2 s_1 s_2^2)^7$$

$$= s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2^2$$

$$= s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2 s_1^2 s_2 \cdot$$

$$s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2^2$$

$$= s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1 s_2 s_1^2 s_2^2 s_1 s_2 s_1 s_2^2 s_1^2 s_2^2$$

$$= s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1^2 s_2 s_1^2 s_2^2 s_1^2 s_2^2$$

$$= s_2^2 s_1^2 s_2 s_1^2 s_2 s_1^2 s_2 s_1^2 s_2^2 = 1.$$

To verify that s_1, s_2 actually generate the simple group of order 168 it may be noted that well-known defining relations of this group are as follows:*

* L. E. Dickson, Linear Groups, 1901, p. 303.

$$T^2 = 1, \quad S^7 = 1, \quad (ST)^3 = 1, \quad (S^4T)^4 = 1.$$

Let $T = s_2s_1s_2s_1$ and $S = (s_1^2s_2s_1s_2^2)^2$. Hence $S^4 = s_1^2s_2^2s_1s_2^2$.

$$(S^4T)^4 = (s_1^2s_2s_1^2s_2s_1)^4 = s_1^2s_2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2s_1 = 1;$$

$$\begin{aligned}(ST)^3 &= s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1 \cdot s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1 \\ &\qquad \cdot s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1 \\ &= s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2^2s_1s_2^2s_1^2s_2s_1^2s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1 \\ &= s_1^2s_2s_1s_2s_1s_2s_1^2s_1s_2s_1s_2s_1s_2^2s_1s_2s_1s_2s_1s_2^2s_1^2 = 1.\end{aligned}$$

It has now been proved that $s_1^2s_2s_1s_2^2$ and $s_2s_1s_2s_1$ generate the simple group of order 168. This is also the group generated by s_1 and s_2 since

$$s_1^2s_2s_1s_2^2 \cdot s_2s_1s_2s_1 = s_1^2s_2s_1^2s_2s_1 = s_2^2s_1s_2^2s_1^2$$

$$s_2s_1s_2s_1 \cdot s_1^2s_2s_1s_2^2 = s_2s_1s_2^2s_1s_2^2 = s_2^2s_1^2s_2s_1^2$$

$$s_1s_2s_1^2s_2 \cdot s_2^2s_1^2s_2s_1^2 = s_1s_2s_1s_2s_1^2$$

$$s_1s_2s_1s_2s_1^2 \cdot s_1^2s_2s_1s_2^2 = s_1s_2s_1s_2s_1s_2s_1s_2 \cdot s_2 = s_2.$$

Since the group generated by $s_1^2s_2s_1s_2^2$ and $s_2s_1s_2s_1$ contains s_2 it contains also $s_1s_2^2s_1^2$ and $s_1^2s_2s_1^2$. The product of these operators is $s_1s_2^2s_1s_2s_1^2 = (s_2^2s_1^2s_2s_1^2)^{-1} \cdot s_1^2$. This proves that the group generated by $s_1^2s_2s_1s_2^2$ and $s_2s_1s_2s_1$ is identical with the group generated by s_1 and s_2. Hence the following theorem has been established: *The simple group of order* 168 *is completely defined by the following equations*:

$$s_1^3 = s_2^3 = (s_1s_2)^4 = (s_1^2s_2)^4 = 1.$$

As is well known, W. Dyck gave in 1882 the following defining equations of this simple group*

$$s_1^7 = 1, \quad s_2^3 = 1, \quad (s_1s_2)^2 = 1, \quad (s_1^5s_2)^4 = 1.$$

§ 4. *The Order of $s_1^2s_2$ is Five.*

If $s_1 = acb \cdot dfe$ and $s_2 = cdf$ it results that $s_1s_2 = adcb \cdot ef$ and $s_1^2s_2 = abdec$. Hence it follows that s_1, s_2 when subjected to the additional condition that $s_1^2s_2$ is of order 5 generate a group G which has the alternating group of degree 6 for a quotient group. L. E. Dickson gave the following defining relations for this quotient group†:

* *Math. Annalen*, vol. 20, p. 41.
† This BULLETIN, vol. 9 (1903), p. 303.

$$t_1^4 = t_2^3 = (t_1^3 t_2 t_1 t_2^2)^2 = (t_1^2 t_2)^4 = 1.$$

Let $t_1 = s_2^2 s_1 s_2^2$ and $t_2 = s_1$. It is easy to see that t_1 and t_2 generate G. In fact, the group generated by t_1 and t_2 contains $(s_2^2 s_1 s_2^2 s_1)^2 = s_1^2 s_2$, and hence it contains s_1 and s_2. The order of t_1 is four since it is the transform of $s_1 s_2$ by s_2. The fact that the order of $t_1^2 t_2$ is also four results from the following equations:

$$s_2^2 s_1 s_2 s_1 s_2^2 s_1 \cdot s_2^2 s_1 s_2 s_1 s_2^2 s_1 \cdot s_2^2 s_1 s_2 s_1 s_2^2 s_1 \cdot s_2^2 s_1 s_2 s_1 s_2^2 s_1$$
$$= s_2^2 s_1 s_2 s_2 s_1 \cdot s_1^2 s_2 s_1^2 s_2 s_1^2 s_2.$$
$$s_2 s_1 s_2^2 s_1 s_2^2 s_1 s_2 s_1 \cdot s_1^2 s_2 s_1^2 s_2 s_1^2 s_2 \cdot s_2 s_1 s_2^2 s_1$$
$$= s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2^2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2.$$
$$s_2^2 s_1 s_2^2 s_1 = s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 \cdot s_1^2 s_2 s_1^2 s_2 \cdot s_2^2 s_1^2 s_2 s_1^2 s_2^2 s_1 s_2^2 s_1$$
$$= s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1 s_2 s_1^2 s_2^2 s_1 s_2^2 s_1 = s_2^2 s_1 s_2^2 s_1^2 \cdot s_1^2 s_2^2 s_1^2 \cdot s_1^2 s_2^2 s_1 s_2^2 s_1$$
$$= 1.$$

It remains to find the order of $t_1^3 t_2 t_1 t_2^2 = s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2$. We shall first prove that the square of this operator is invariant under each of the operators s_1 and s_2. The following equations exhibit the fact that this square is invariant under s_2:

$$s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2$$
$$= s_1^2 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2$$
$$= s_2 s_1 s_2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2$$
$$= s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2$$
$$= s_2 s_1^2 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2$$
$$= s_2 s_1^2 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2$$
$$= s_2 s_1^2 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2$$
$$= s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1^2 s_2$$
$$= s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2^2 s_1

The transform of this square by s_1 is

$$s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1^2s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1s_2^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1^2s_2s_1s_2^2s_1^2.$$

Having proved that this square is invariant under s_1 and s_2 we proceed to prove that the cube of this square is identity:

$$s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2$$

$$\cdot s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2$$

$$= s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2$$

$$\cdot s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2$$

$$= s_2^2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2$$

$$= s_1^2s_2s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2^2s_1s_2^2s_1^2s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1s_2^2s_1^2s_2^2s_1s_2^2s_1^2s_2^2s_1s_2^2s_1^2s_2s_1^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1^2s_2^2s_1s_2^2s_1^2s_2s_1^2$$

$$= s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1^2s_2^2s_1^2$$

$$s_1s_2^2s_1^2s_2s_1^2s_2^2s_1s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2$$

$$\cdot s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2$$

$$= s_2s_1^2s_2s_1^2s_2^2s_1s_2s_1^2s_2s_1s_2^2s_1s_2s_1s_2s_1s_2^2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1$$

$$= s_1s_2^2s_1s_2^2s_1s_2s_1s_2s_1^2s_2s_1s_2^2s_1s_2s_1^2s_2s_1^2s_2s_1s_2^2s_1^2s_2s_1^2s_2s_1s_2^2s_1$$

$$= s_1^2s_2^2s_1s_2s_1^2s_2^2s_1s_2s_1s_2s_1^2s_2s_1s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1^2s_2s_1s_2^2$$

$$= s_1^2s_2^2s_1s_2s_1^2s_2s_1^2s_2^2s_1s_2s_1s_2s_1^2s_2s_1s_2^2s_1^2s_2^2s_1^2s_2s_1^2s_2s_1s_2^2$$

$$= s_1^2s_2^2s_1s_2s_1^2s_2s_1^2s_2s_1^2s_2^2s_1s_2s_1s_2s_1^2s_2s_1^2s_2^2s_1^2s_2s_1^2s_2s_1s_2^2$$

$$= s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 = 1.$$

Since $(t_1^3t_2t_1t_2^2)^2$ is invariant under G and the order of this invariant operator is a divisor of 3, it results that the order g of G must be a divisor of 1,080. If $g = 1{,}080$, G can be represented as a transitive substitution group of degree 18 since it contains the icosahedral group, as results from the fact that

$$s_1^2s_2 \quad \text{and} \quad s_1s_2s_1s_2$$

are two operators of orders 5 and 2 respectively whose product is of order 3, since

$$s_1^2s_2s_1s_2s_1s_2s_1^2s_2s_1s_2s_1s_2s_1^2s_2s_1s_2s_1s_2 = s_1s_2^2s_1^6s_2^2s_1^6s_2^2s_1^2 = 1.$$

The subgroup of order 60 composed of all the substitutions which omit one letter of this transitive group is of degree 15 since G contains an invariant operator of order 3, and the six systems of intransitivity of the two invariant substitutions of order 3 of this transitive group are permuted according to the alternating group of degree 6. If this subgroup is transitive, it is completely determined since the icosahedral group can be represented in essentially only one way as a transitive substitution group of degree 15. That is, all such possible representations are conjugate substitution groups. Moreover, this transitive subgroup determines five cycles of the two invariant substitutions of order 3 contained in G.

If we assume that this subgroup is generated by the following substitutions:

$$abc\cdot dkf\cdot enh\cdot goi\cdot jlm, \qquad al\cdot bn\cdot cf\cdot do\cdot gm\cdot hi,$$

then the invariant substitutions of the substitution group of degree 18 which is simply isomorphic with G may be assumed to be

$$abc\cdot dmh\cdot ekj\cdot fln\cdot gio\cdot pqr$$

$$acb\cdot dhm\cdot ekj\cdot fnl\cdot goi\cdot prq.$$

An additional generator of this substitution group is of order 4 and has for its square any one of the substitutions of order 2 contained in it, since all of these substitutions of order 2 are conjugate under this substitution group. As this substitution of order 4 is also commutative with the given invariant substitutions of order 3 and must involve three cycles of order 4 and three transpositions, it must be one of three substitutions. Assuming that its square is

$$dj\cdot em\cdot fi\cdot gn\cdot hk\cdot lo,$$

these three substitutions are as follows:

$$ap\cdot bq\cdot cr\cdot dfji\cdot mleo\cdot hnkg$$

$$ap\cdot bq\cdot cr\cdot dljo\cdot mneg\cdot hfki$$

$$ap\cdot bq\cdot cr\cdot dnjg\cdot mfei\cdot hlko.$$

It is not difficult to verify that the first two of these substitutions together with the above group of order 60 and degree 15 give rise to substitutions which could not appear in the group under consideration. On the other hand, the last one of these three substitutions together with the group of order 60 actually generates a group of order 1,080 and of degree 18. This group is generated by the following substitutions:

$$s_1 = alk\cdot bnj\cdot cfe\cdot dmh\cdot goi,$$

$$s_2 = apk\cdot bqj\cdot cre\cdot dif\cdot gnh\cdot lmo.$$

The fact that these substitutions satisfy the conditions imposed on s_1 and s_2 results from the following equations:

$$s_1s_2 = amgl\cdot bhin\cdot cdof\cdot er\cdot jq\cdot kp,$$

$$s_1^2s_2 = dgfre\cdot honqj\cdot ilpkm.$$

The given substitutions satisfy also the conditions imposed on t_1 and t_2, since

$$t_1 = ak\cdot bj\cdot ce\cdot dril\cdot fhqg\cdot mpon,$$

$$t_1^2t_2 = alrk\cdot bnpj\cdot cfql\cdot dg\cdot ho\cdot im,$$

$$t_1^3t_2t_1t_2^2 = alcfbn\cdot dkhemj\cdot gio\cdot pqr.$$

The group generated by s_1 and s_2 contains

$$(s_1s_2s_1)^2(s_1s_2)^2s_1^2s_2 = dhm\cdot eno\cdot fgk\cdot \breve{c}jl\cdot pqr.$$

As the product of this substitution of order 3 and $s_1^2s_2$ is of order 2, these two substitutions generate a transitive group of order 60 and of degree 15. It therefore results that s_1 and s_2 generate a transitive group of degree 18 whose order cannot be less than 1,080, and from the given abstract properties of s_1 and s_2 it follows that this order cannot exceed 1,080. It has therefore been proved that the G is actually of order 1,080.

In the given group of order 60 each substitution of order 3 is transformed into itself by its own powers only. Hence in the group of order 1,080 such a substitution is transformed into itself by only 9 substitutions. The Sylow subgroups of order 27 contained in this group must therefore be non-abelian. Hence it results that G does not contain a subgroup which is simply isomorphic with the alternating group of degree 6.

It may be noted that to every operator in the central quotient group there corresponds at least one operator of G whose order is equal to the order of the operator of this quotient group, and yet G does not involve any subgroup which is simply isomorphic with the central quotient group. In this respect the present group differs from the group of order 120 which has for its central quotient group the icosohedral group but does not contain the latter as a subgroup. In this group of order 120, operators of order 4 correspond to operators of order 2 in the central quotient group. The Sylow subgroups of order 8 contained in G are separately simply isomorphic with the octic group, just as in the simple group of order 360, while the Sylow subgroups of order 27 are separately simply isomorphic with the non-abelian group of this order which involves no operator of order 9. In particular, the following theorem has been established: *There are two and only two groups, besides identity, which are generated by s_1 and s_2 when these two operators satisfy the conditions*

$$s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1^2 s_2)^5 = 1.$$

One of these is the simple group of order 360 *and the other is a group of order* 1,080 *whose central quotient group is this simple group but which does not involve this simple group as a subgroup.*

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NOTE ON RIEMANN SPACES.

BY DR. JAMES W. ALEXANDER.

(Read before the American Mathematical Society February 28, 1920.)

1. It is proposed to establish the theorem that every closed orientable n-dimensional manifold can be represented on an n-dimensional hypersphere as a Riemann space, or generalized Riemann surface. The argument will be carried through explicitly for the case $n = 3$ only, since the extension to higher dimensions is perfectly automatic and requires scarcely more than the modification of a few subscripts.

2. We know that a 3-dimensional manifold can always be built up out of the points and boundary points of a finite number of tetrahedral regions by suitably matching together in pairs the triangular faces of the bounding tetrahedra. Let $A_1, A_2, \cdots, A_k$ be the points of the manifold which correspond to the vertices of the tetrahedra. Then it can always be so arranged that the four points A_{i1}, A_{i2}, A_{i3}, and A_{i4} corresponding to the vertices of a tetrahedron are distinct, in which case, the bounded tetrahedral region may be designated by the symbol $|A_{i1}A_{i2}A_{i3}A_{i4}|$, where the ordering of the letters A_{ij} is immaterial.

We shall say that every permutation of the letters in the symbol for a tetrahedral region determines a sense on the region and that two permutations determine the same or opposite senses according as they differ by an even or an odd number of transpositions. A tetrahedral region becomes sensed, or oriented, by association with one of the permutations and is then conveniently designated either by the symbol for the sense-giving permutation or by the symbol for any other permutation taken with a plus or minus sign according as the second permutation differs from the first by an even or an odd number of transpositions. Thus, we have

$$A_1A_2A_3A_4 = -A_2A_3A_4A_1 = A_3A_4A_1A_2 = \cdots,$$

but

$$A_1A_2A_3A_4 \neq A_2A_3A_4A_1, \text{ etc.}$$

3. Now, consider two tetrahedral regions $|AA_1A_2A_3|$ and $|A'A_1A_2A_3|$ with a face $A_1A_2A_3$ in common. If senses be

assigned to these regions, the given senses will be said to be consistent at the face $A_1A_2A_3$ if and only if we have one of the following combinations

$$(1)\qquad \begin{matrix} AA_1A_2A_3 & \text{and} & -A'A_1A_2A_3 & \text{or} \\ -AA_1A_2A_3 & \text{and} & A'A_1A_2A_3. & \end{matrix}$$

The manifold is said to be orientable if senses can be assigned to all the tetrahedral regions in such a way as to be consistent at every face. The property of being or not being orientable is an invariant of the manifold and does not depend on the selection of the set of tetrahedral regions.* From now on, we shall confine the discussion to orientable manifolds and shall assume that the tetrahedral regions have all been consistently sensed.

4. Now suppose that in the space of inversion (with a single point at infinity, and therefore like the hypersphere topologically) we set up a system of axes and select k finite points $P_i(x_i, y_i, z_i)$, $i=1, 2, \cdots, k$, one for each of the points A_i, and such that no four are coplanar. Then, any four of the points, such as P_1, P_2, P_3 and P_4, determine a tetrahedron which subdivides the space into two tetrahedral regions, one of which does not contain the point at infinity. We designate the latter region by any permutation $P_{i1}P_{i2}P_{i3}P_{i4}$ of the points P_1, P_2, P_3, P_4 such that

$$\Delta(P_{i1}P_{i2}P_{i3}P_{i4}) = \begin{vmatrix} x_{i1} & x_{i2} & x_{i3} & x_{i4} \\ y_{i1} & y_{i2} & y_{i3} & y_{i4} \\ z_{i1} & z_{i2} & z_{i3} & z_{i4} \\ 1 & 1 & 1 & 1 \end{vmatrix} > 0,$$

and the one which contains the point at infinity by any permutation $P_{j1}P_{j2}P_{j3}P_{j4}$ such that

$$\Delta(P_{j1}P_{j2}P_{j3}P_{j4}) < 0.$$

With this notation, the desired Riemann space is obtained at once by merely mapping each sensed region $A_{s1}A_{s2}A_{s3}A_{s4}$ of the manifold on the region $P_{s1}P_{s2}P_{s3}P_{s4}$ of the space of inversion, in such a way, of course, that the vertex A_{st} always goes over into the point P_{st} and that the images of the boundaries of various tetrahedral regions join on to one another properly along the images of their faces and edges.

* In the Poincaré terminology, a necessary and sufficient condition that a manifold be orientable is that it have no coefficient of torsion of order n.

The proof that the mapping just defined gives the desired Riemann space rests on the fact that two determinants

$$\Delta(PP_1P_2P_3) \quad \text{and} \quad -\Delta(P'P_1P_2P_3)$$

have the same or opposite signs according as the points P and P' lie on opposite sides of the plane $P_1P_2P_3$ or on the same side. It therefore follows that for two contiguous regions

$$(2) \qquad \begin{matrix} PP_1P_2P_3 & \text{and} & -P'P_1P_2P_3 & \text{or} \\ -PP_1P_2P_3 & \text{and} & P'P_1P_2P_3 & \end{matrix}$$

the points of one region in the neighborhood of a point of the face $P_1P_2P_3$ are on opposite sides of the face from those of the other region. Hence, by comparison of (1) and (2), it is seen that the mapping is uniform not only in the neighborhood of points of the manifold within a tetrahedral region but also of points on a bounding face. Therefore, the only singularities in the mapping correspond to vertices A_i or edges A_iA_j. Furthermore, the mapping is everywhere r to 1 on the space of inversion, except perhaps at the points P_i or along the edges P_iP_j, because any two points, B and C, not on a line P_iP_j can be joined by a path p which does not meet a line P_iP_j. Therefore, if the point B corresponds to n points B_1, B_2, $\cdots$, B_n of the manifold, the path p corresponds to n non-intersecting paths leading from the B_i's to an equal number of points C_i.

5. In the 3-dimensional case, a Riemann space obtained by the above construction contains, in general, a network of branch lines at each of which two or more sheets coalesce. It is easy to show that, without modifying the topology of the space, the branch system may be replaced by a set of simple, non-intersecting, closed curves such that only two sheets come together at a curve. The curves may, however, be knotted and linking.

Three-dimensional Riemann spaces have been discussed by Heegaard* and Tietze,† but neither of these mathematicians seems to have been aware of their complete generality.

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* Heegaard, Inaugural Dissertation, Copenhagen, 1898. Also: *Bulletin Société math. de France*, vol. 44 (1916) p. 161.

† Tietze, *Monatshefte für Mathematik und Physik*, vol. 19 (1908), p. 1. Cf. remark on p. 104.

SHORTER NOTICES.

Junior High School Mathematics. By GEORGE WENTWORTH, DAVID EUGENE SMITH, AND JOSEPH CLIFTON BROWN. Book 3; 282 pages, price 96 cents. Boston, Ginn, 1918.

FOR many years past, the unification of elementary mathematics has been a constant subject of discussion. Until recently, however, very little progress has been made in the direction of a practical solution of this problem. Only in vocational education has there been any definite attempt to modernize and coordinate instruction in mathematics, and in this field results are necessarily of somewhat limited application. Everyone interested in educational affairs must recognize the necessity of keeping pace with the growing needs and opportunities of our time. The members of the American Mathematical Society as well as teachers of high school mathematics should therefore feel indebted to the authors of the work under review for having so effectively modernized high school mathematics, especially geometry, and for the attractive and teachable form in which the results are here presented.

The two books of the series which precede the one under review were intended for the seventh and eighth grades, and include arithmetic, the elements of algebra, and intuitive geometry. Book 3 is based on this preliminary work, and includes algebra, the elements of plane trigonometry, and demonstrative geometry.

In particular, the first section, on algebra, includes the four fundamental processes applied to monomials, polynomials and fractions, the solution of simple and quadratic equations, and of simultaneous linear equations. Algebra is here presented as a universal shorthand, and approached from the practical standpoint of the formula. A pedagogical feature of special importance is the clear and concise treatment, and the use of the unit page, no topic overrunning the page, making each page complete in itself.

The section on plane trigonometry defines the trigonometric functions, and illustrates their practical application to the solution of right triangles. The importance of this section, of course, does not consist in its mathematical content, but

in the stimulus it gives to effort by opening a new and practical field of applications intelligible to every child. As this involves both geometrical and arithmetical magnitudes, it also naturally serves as the connecting link between what precedes on algebra and what follows on geometry.

The concluding section on demonstrative geometry, however, is the one of greatest pedagogical importance, as it makes a radical departure from the canon of Euclid. In the first place, there is no attempt to base the subject on a minimum number of axioms. For instance, it is assumed that at a point on a straight line one perpendicular and only one can be erected. Moreover, angles and triangles are compared and classified, and several problems in construction are explained before there is any attempt at formal demonstration. The axioms are simply the four relating to equals combined with equals, and are all stated on one unit page and illustrated numerically. The postulates are ten in number, and are also listed on one page, and include many statements commonly required to be proved. This variation from Euclid is extremely important, as children frequently get a lasting distaste for, and misconception of, geometry by being required to prove statements which are perfectly obvious, such as that a circle is bisected by a diameter. The book also calls attention to the fact that the axioms here introduced were previously applied in algebra, thereby impressing the child with the important fact that these two subjects have a common mathematical basis.

A very clear explanation is then given of the nature of a proof, and how an inference may be examined and proved. This is an especially meritorious feature, as there are few teachers capable of presenting the true nature of geometrical reasoning in such a clear and concise manner. The nature of an inference is further illustrated by comparison of a geometrical inference with one based on medical diagnosis, or on legal evidence. This is certainly calculated to give training in correct thinking, generally recognized to be so much needed at the present time.

Other features of the book show painstaking care and unusual teaching ability, such as the selection of interesting and practical problem material and the specially prepared set of full page perspective drawings, relating the plane figures of geometry to our three-dimensional world of experience.

The work certainly marks the greatest advance in the pedagogy of elementary mathematics which has yet appeared, and is deserving of a wide adoption as a text.

S. E. SLOCUM.

Business Arithmetic. By C. W. SUTTON and N. J. LENNES. Allyn and Bacon, 1918.

THIS book begins in the most elementary manner with the most elementary operations of arithmetic and covers in great detail their applications to almost every kind of a business transaction which one could imagine which calls for such elementary operations. A printed list of topics which are considered would be far too long to give here. So much information of a business nature is given and given so carefully that the book should prove valuable to any one, particularly a business man, as a book of reference; it is veritably a compendium of business knowledge. However, as a textbook, it violates the well-developed belief that better results are obtained by presenting relatively few fundamental principles and facts and then devoting the rest of the time to stimulating independent thought, rather than by the statement and explanation of a multitude of fairly independent principles and facts. It may be that business arithmetic is an exception. Only those who are familiar with the great number of terms and expressions involved can appreciate the difficulties of attempting to treat the subject in accordance with the theory advanced above.

There is a multitude of problems—there are over one hundred problems on some pages—but the problems are so elementary and frequently so similar that it seems at times that sufficient drill could be obtained by a smaller number and valuable time thus gained.

It is the reviewer's belief that any textbook on elementary mathematics—and especially one on business arithmetic—should include at least a simple statement or discussion of some of the rules for numerical computation. It is with considerable misgivings that we note (in Art. 147) the instructions given the student in adding the numbers 12., 1.49, .978, 640.2, 4.904, .007, which will surely serve to authorize him to ignore the adopted convention followed in writing numbers to indicate the degree of the accuracy. Twelve problems follow on page 115 establishing firmly this dangerous fault which is already all too prevalent.

The book is carefully written and printed and is excellent for purposes of a thorough drill in business principles, but it is not conducive to rigorous and independent thinking. It would probably prove very useful as a textbook in a business school.

C. H. FORSYTH.

Empirical Formulas. By THEODORE R. RUNNING. No. 19 of Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. New York, Wiley, 1917. 144 pp.

ONE of the problems of the engineer, chemist, physicist or statistician is the finding of a formula, simple as possible, by means of which an approximate value of one variable may be computed from a given value of another variable. The volume under review is concerned with this indeterminate problem, which is not necessarily the problem of determining the physical law connecting the variables in question. In the first five chapters, which make up the principal part of the book, twenty different forms of equations are taken up, tests are given to show when each type of equation may be suitable for given data, and methods for determining constants are discussed and in many cases illustrated by a concrete example worked out in detail.

For example, one form of equation discussed, No. XVI, is

$$(x + a)(y + b) = c.$$

To test a given set of data thought to follow this law, we plot

$$x - x_k, \qquad \frac{x - x_k}{y - y_k},$$

where x_k, y_k are any two corresponding values from the data. If these points lie on a straight line, then it may be assumed that x and y are related as above and we may proceed to find the constants. To illustrate this case the results of a set of experiments giving the potential difference and the current in an electric arc are used and the problem worked out in detail.

Included in these twenty types of equations is the Fourier's series with a limited number of terms to which a whole chapter is devoted. Twenty figures in an appendix show some of the forms of curves belonging to each equation considered. The

last three chapters are on the method of least squares, interpolation and numerical integration.

The book is clear and very easy reading even for the undergraduate. There is perhaps too much detail. The presentation on the page is pleasing and there are few errors. The few found when the book was used in a class were not important. The reviewer regrets that the author stopped short of a discussion of the method of moments and of frequency curves. These however could be included in a companion volume.

A. R. CRATHORNE.

NOTES.

THE American association for the advancement of science will hold its seventy-first meeting at Chicago from December 27, 1920, to January 1, 1921. What has been heretofore Section A, mathematics and astronomy, of the Association has been divided into two sections, A, mathematics, and D, astronomy. Of Section A, Professor D. R. CURTISS has been elected vice-president and Professor W. H. ROEVER secretary, Professors DUNHAM JACKSON, A. D. PITCHER, G. A. BLISS, J. M. PAGE, and H. L. RIETZ members of the sectional committee, Professor G. A. MILLER member of the council, and Professor E. V. HUNTINGTON member of the general committee. Professor JOEL STEBBINS has been elected vice-president and Professor F. R. MOULTON secretary of Section D.

THE British association for the advancement of science will meet at Cardiff, beginning August 24, under the presidency of Professor W. A. HERDMAN. Professor A. S. EDDINGTON has been appointed president of Section A, mathematics and physics.

THE Mathematical society of Greece announces the publication of an official journal, the *Bulletin of the Mathematical Society of Greece*, of which the first volume appeared in 1919. Articles for insertion in the *Bulletin* should be addressed to Professor NILOS SAKELLARIOU, Rue Asklipiou 96, Athens.

The following university courses in mathematics are announced:

COLUMBIA UNIVERSITY (summer session, July 6 to August 13).—By Professor EDWARD KASNER: Fundamental concepts of modern mathematics, five hours; Geometric transformations and groups, five hours.—By Professor W. B. FITE: Differential equations, five hours.—By Dr. G. A. PFEIFFER: Introduction to higher algebra, five hours.

COLUMBIA UNIVERSITY (academic year 1920–1921).—By Professor T. S. FISKE: Theory of functions, four hours.—By Professor F. N. COLE: Algebra, four hours.—By Professor D. E. SMITH: History of mathematics, two hours; Practicum in the history of mathematics, four hours.—By Professor EDWARD KASNER: Seminar in differential geometry, two hours.—By Professor W. B. FITE: Differential equations, three hours.—By Dr. J. F. RITT: Selected topics in the theory of functions of a complex variable, three hours.—By Dr. G. A. PFEIFFER: Analysis situs and applications, three hours.

PROFESSOR M. BORN, of the University of Berlin, has been appointed professor of theoretical physics at the University of Frankfurt, as successor to Professor M. VON LAUE, who has been appointed to the same position at the University of Berlin.

PROFESSOR M. KRAUSE has been elected rector of the Dresden technical school.

DR. E. TREFFTZ, of the technical school at Aachen, has been promoted to a full professorship of mathematics.

DR. A. VON BRUNN has been promoted to a professorship at the Danzig technical school.

DR. H. FALCKENBERG, of the technical school at Braunschweig, has been admitted as privat docent at the University of Königsberg.

PROFESSOR W. OLBRICH has been appointed associate professor of descriptive geometry at the Vienna agricultural school.

ASSOCIATE professor A. TAUBER, of the University of Vienna, has been promoted to a full professorship of mathematics.

ACTUARY ARNFINN PALMSTRÖM has been appointed professor of insurance at the University of Christiania.

PROFESSOR L. BIANCHI, of the University of Pisa, and Sir JOSEPH LARMOR, of Cambridge University, have been elected corresponding members of the Paris academy of sciences, in the section of geometry.

MR. S. POLLARD, of Trinity College, has been awarded a Smith's prize for 1920 for his paper on "The Stieltjes integral and its generalizations."

PROFESSOR J. HADAMARD delivered a course of lectures at Yale University, beginning April 30, on "Some topics in linear partial differential equations."

AT Brown University, associate professor H. P. MANNING has retired from active service, after twenty-nine consecutive years as teacher in the department of mathematics. Dr. R. F. BORDEN, of the University of Illinois, has been appointed instructor in mathematics.

PROFESSOR G. D. BIRKHOFF, of Harvard University, has been elected a member of the Danish academy of sciences.

PROFESSOR O. D. KELLOGG has resigned his position in the University of Missouri and has become permanently connected with Harvard University as associate professor of mathematics.

PROFESSOR J. E. ROWE, of Pennsylvania State College, has resigned to accept an appointment as mathematics and dynamics expert in the ordnance bureau of the war department. He is serving as chief ballistician at the Aberdeen Proving Ground.

PROFESSOR W. L. G. WILLIAMS, of William and Mary College, has been appointed instructor in mathematics at Cornell University.

PROFESSOR CHESTER SNOW, of the department of mathematics at the University of Idaho, has resigned to accept a position as physicist in the bureau of standards at Washington.

PROFESSOR A. D. BUTTERFIELD, of the department of mathematics at the Worcester Polytechnic Institute, has resigned to become educational director for the Norton Company.

PROFESSOR WALDEMAR VOIGT, of the University of Göttingen, died December 13, 1919, at the age of sixty-nine years.

PROFESSOR MORITZ CANTOR, of the University of Heidelberg, died April 10, 1920, at the age of ninety-one years.

PROFESSOR G. E. FISHER, of the University of Pennsylvania, died March 28, 1920, at the age of fifty-seven years, after thirty-one years service in the department of mathematics. Professor Fisher became a member of the American Mathematical Society in 1891.

CORRECTION. On page 331 of the April BULLETIN the date of the opening of the first term of the University of Chicago summer session should be June 21.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOREL (E.). See VALLÉE POUSSIN (C. DE LA).

BREUER (S.). Ueber die irreduktiblen auflösbaren trinomischen Gleichungen fünften Grades. (Diss.) Frankfurt a. M., 1918.

BROUWER (L. E. J.). Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. 2ter Teil: Theorie der Punktmengen. Amsterdam, Johannes Müller, 1919. M. 1.60

BÜRKLEN (O.). Aufgaben zur analytischen Geometrie des Raumes. 2te Auflage. Berlin, 1918. M. 1.25

DECKERT (A.). Analytische Geometrie. Breslau, 1918. M. 7.50

DURELL (C. V.). Modern geometry: The straight line and circle. London, Macmillan, 1920. 10 + 145 pp. 6s.

FELIX KLEIN zur Feier seines siebzigsten Geburtstages. Sonderheft: Die Naturwissenschaften. 25. April 1919. Berlin, Springer, 1919.

HAMMERSTEIN (A.). Zwei Beiträge zur Zahlentheorie. (Diss.) Göttingen, 1919.

HATTON (J. L. S.). The theory of the imaginary in geometry together with the trigonometry of the imaginary. Cambridge, University Press, 1920. 8vo. 8 + 216 pp. 18s.

HELM (G.). Die Grundlehren der höheren Mathematik. Zum Gebrauch bei Anwendungen und Wiederholungen zusammengestellt. Neue, verbesserte Ausgabe. Leipzig, 1919. M. 20.40

HENKLER (P.). Einführung in mathematisches Denken. Berlin, Union, 1919. 55 pp. M. 2.40

HÖLDER (O.). Die Mathematik im Verhältnis zu den anderen Wissenschaften. Leipzig, 1918.

JUNKER (F.). Höhere Analysis. 2ter Teil: Integralrechnung. 3te, verbesserte Auflage. Berlin, 1919. M. 1.25

——. Repetitorium und Aufgabensammlung zur Integralrechnung. 3te, verbesserte Auflage. Berlin, 1919. M. 1.25

KLEIN (F.). See FELIX KLEIN.

KOENIGSBERGER (L.). Mein Leben. Heidelberg, Carl Winter, 1919. M. 6.00

KOWALEWSKI (G.). Grundzüge der Differential- und Integralrechnung. 2te, verbesserte Auflage. Leipzig, 1919. 416 pp. M. 12.00

LIETZMANN (W.). Methodik des mathematischen Unterrichts. 1ter Teil: Organization, allgemeine Methode und Technik des Unterrichts. Leipzig, Quelle und Meyer, 1919.

NEDER (L.). Zur Konvergenz der trigonometrischen Reihen, einschliesslich der Potenzreihen auf dem Konvergenzkreise. (Diss.) Göttingen, 1919.

NEPPI (M. A.). Sul luogo geometrico dei punti del piano di un triangolo aventi i triangoli podari ortogonali equivalenti fra loro; inviluppi della retta di Simson ottenuta con qualunque angolo. Alessandria, Gazzotti, 1919. 8vo. 12 pp.

SIMON (M.). Analytische Geometrie des Raumes. 3te Auflage. Neudruck. Berlin, 1919. M. 1.25

SPIEWECK (B.). Genauere Untersuchung der Kurven $A\kappa + B\tau = C\tau/\kappa$. (Diss.) Halle, 1919.

VALLÉE POUSSIN (C. DE LA). Leçons sur l'approximation des fonctions d'une variable réelle professées à la Sorbonne. (Collection de Monographies sur la Théorie des Fonctions publiée sous la direction de M. Émile Borel.) Paris, Gauthier-Villars, 1919. 8vo. 6 + 150 pp. Fr. 12.00

WIDMER (A.). Ueber die Anzahl der Lösungen gewisser Kongruenzen nach einem Primzahlmodul. (Diss., Technische Hochschule, Zürich.) 1919.

II. ELEMENTARY MATHEMATICS.

AMICI (N.) e MEDICI (S.). Lezioni di aritmetica ed algebra. Volume 2, per il secondo corso dell'istituto tecnico. 2a edizione. Rocca S. Casciano, L. Cappelli, 1919. 8vo. 268 pp. L. 6.00

ARNOLD (E. E.). See DURELL (F.).

COMSTOCK (C. E.). See SYKES (M.).

DURELL (F.) and ARNOLD (E. E.). A first book in algebra. Chicago, Merrill, 1919. 12mo. 5 + 325 + 39 pp.

GUGLE (M.). Modern junior mathematics. New York, Gregg Publishing Company, 1920. 12mo. Book 1, 9 + 222 pp.; book 2, 11 + 239 pp.

MARTINI (Z. A.). Trattato di algebra elementare, ad uso degli istituti tecnici. 8a edizione, riveduta. Livorno, R. Giusti, 1920. 8vo. 15 + 379 pp. L. 6.50

MEDICI (S.). See AMICI (N.).

PRÉ (L.). Méthode rapide pour le calcul des intérêts et des déductions. Paris, Gauthier-Villars, 1920. 8vo. 134 pp. Fr. 7.50

ROCQUES DESVALLÉES (H.). Tables logarithmiques et trigonométriques à quatre décimales (argument en arc et en temps) et tables à trois décimales, à l'usage des physiciens et des navigateurs. Paris, Gauthier-Villars, 1920. 8vo. 24 + 72 pp. Fr. 6.00

SYKES (M.) and COMSTOCK (C. E.). Plane geometry. Chicago, Rand McNally, 1918. 12mo. 12 + 322 pp.

III. APPLIED MATHEMATICS.

ARDUNIO (E.). Statistica. 5a edizione, riveduta ed ampliata. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1920. 16mo. 8 + 118 pp. L. 1.80

BARLOW (G.). See POYNTING (J. H.).

BEAUVAIS (P.). See NÈGRE (F.).

BRÄUER (P.). Ionentheorie. (Mathematisch-physikalische Bibliothek, Band 38.) Leipzig, Teubner, 1919. M. 1.00

CAMPBELL (N. R.). La théorie électrique moderne. Théorie électronique. Traduit sur la 2e édition anglaise par A. Corvisy. Paris, Hermann, 1919. 464 pp. Fr. 18.00

CAPART (G.). See WIEWEGER (H.).

CORVISY (A.). See CAMPBELL (N. R.).

CRANZ (H.). See HUGERSHOFF (R.).

EASTWOOD (G. S.) and FIELDEN (J. R.). An algebra for engineering students. London, Arnold, 1920. 7 + 199 + 15 pp. 7s. 6d.

FIELDEN (J. R.). See EASTWOOD (G. S.).

GRÜBLER (M.). Lehrbuch der technischen Mechanik. 1ter Band: Bewegungslehre. Berlin, 1919. M. 8.00

HAUBER (W.). Die Grundlehren der Statik starrer Körper. 5ter Neudruck. Berlin, 1919. M. 1.25

HAUSSNER (R.). Darstellende Geometrie. 2ter Teil: Perspektive ebener Gebilde; Kegelschnitte. 2te Auflage. Berlin, 1919. M. 1.25

HENDERSON (R.) and SHEPPARD (H. N.). Graduation of mortality and other tables. (Actuarial Studies, No. 4.) New York, Actuarial Society of America, 1919. 6 + 82 pp. $1.25

HUGERSHOFF (R.) und CRANZ (H.). Grundlagen der Photogrammetrie aus Luftfahrzeugen. Stuttgart, 1919. M. 13.20

LORENZ (H.). Einführung in die Technik. Leipzig, Teubner, 1919. 94 pp. M. 2.15

LOVE (A. E. H.). A treatise on the mathematical theory of elasticity. Cambridge, University Press, 1920. Royal 8vo. 18 + 624 pp. 37s. 6d.

LOVETT (W. J.). Applied naval architecture. London, Longmans, 1920. 8vo.

MAACK (F.). Raumschach. Einführung in die Spielpraxis. Hamburg, F. Maack, 1919.

MACKENZIE (I. S. F.). A night raid into space: the story of the heavens told in simple words. London, H. Hardingham, 1920. 143 pp. 2s. 6d.

MATTHEWS (R. B.). The aviation pocket-book for 1919–20. A compendium of modern practice and a collection of useful notes, formulæ, rules, tables, and data relating to aeronautics. 7th edition, revised and enlarged. London, C. Lockwood and Son, 1919. 24 + 536 pp. 12s. 6d.

MERIZZI (C.). Lezioni di geometria descrittiva. 2a edizione, coretta. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1920. 16mo. 7 + 202 pp. L. 2.70

MIE (G.). Das Wesen der Materie. I: Moleküle und Atome. 4te Auflage. (Aus Natur und Geisteswelt, Nr. 58.) Leipzig, Teubner, 1919. 122 pp.

MÖLLER (J.). Nautik. 2te Auflage. (Aus Natur und Geisteswelt, Nr. 255.) Leipzig, Teubner, 1919. 116 pp.

MOSS (H.). See WATSON (W.).

MOUNIER (J.). Les graphiques du patron donnant une solution immédiate approchée de tous problèmes de banque, intérêts, escompte, prix de vente, placements. Paris, Gauthier-Villars, 1920. 4to. 24 pp. Fr. 9.00

MOUREU (C.). See THOMSON (J. J.).

NÈGRE (F.) et BEAUVAIS (P.). Calcul, construction et essais d'une dynamo à courant continu. Paris et Liège, 1914 (paru en 1919). 8vo. 385 pp. Fr. 22.50

PLANCK (M.). Einführung in die Mechanik deformierbarer Körper. Zum Gebrauch bei Vorträgen sowie Selbstunterricht. Leipzig, 1919. M. 9.50

POYNTING (J. H.). Collected scientific papers by J. H. Poynting. Edited by G. A. Shakespear and G. Barlow. Cambridge, University Press, 1920. Royal 8vo. 32 + 768 pp. 37s. 6d.

RUSCH (F.). Beobachtung des Himmels mit einfachen Instrumenten. 2te Auflage. (Mathematisch-physikalische Bibliothek, Band 14.) Leipzig, Teubner, 1919. 51 pp.

SCHOUTEN (J. A.). Die direkte Analysis zur neueren Relativitätstheorie. Amsterdam, 1919. M. 3.20

SCHUDEISKY (A.). Geometrisches Zeichnen. (Aus Natur und Geisteswelt, Nr. 568.) Leipzig, Teubner, 1919. 99 pp.

SCHURITZ (H.). Die Perspektive in der Kunst Dürers. (Diss., Technische Hochschule, Darmstadt.) Frankfurt a. M., H. Keller, 1919.

SHAKESPEAR (G. A.). See POYNTING (J. H.).

SHEPPARD (H. N.). See HENDERSON (R.).

SUMNER (P. H.). The design and stability of stream-line kite balloons, with useful tables, aeronautical and mechanical formulæ. London, C. Lockwood and Son, 1920. 8 + 146 pp. 10s. 6d.

TANCOCK (E. O.). Elements of descriptive astronomy. 2d edition. Oxford, Clarendon Press, 1920. 158 pp. 3s.

THOMSON (J. J.). La théorie atomique. Traduit de l'anglais par C. Moureu. Paris, Gauthier-Villars, 1919. 16mo. 6 + 58 pp. Fr. 2.00

WATSON (W.) and MOSS (H.). A text-book of physics. Including a collection of examples and questions. London, Longmans, 1920. 8vo. 17s. 6d.

WIEWEGER (H.). Recueil de problèmes avec solutions sur l'électricité et ses applications pratiques. Traduit par G. Capart. 3e édition, nouveau tirage. 1919. 8vo. 16 + 400 pp. Fr. 22.50

WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 1ter Band. Berlin, 1919. M. 14.00